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On the stability of stratified viscous plane Couette flow. Part 2. Variable buoyancy frequency

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A statically stable stratification with buoyancy frequency $N^2(z) = z^2$ is found to cause large changes in the modal structure for viscous plane Couette flow (as compared with the case when $N^2(z) = 1$) and it also has a strongly destabilizing effect on the flow. On minimizing with respect to both wavenumber and Richardson number, it is found that the flow is unstable if the Reynolds number is greater than about 183. A study of the Reynolds stress and vertical buoyancy flux shows that there is a large transfer of energy from the basic flow to the velocity disturbances and this is consistent with such a surprisingly low value of the minimum critical Reynolds number.

1. Introduction

In the preceding paper (Davey & Reid 1977; hereinafter referred to as part 1) we considered the modal structure of stratified plane Couette flow with a constant buoyancy frequency, and in this paper we wish to consider the same flow with a variable buoyancy frequency given by $N^2(z) = z^2$. This is a case for which the local Richardson number does *not* exceed $\frac{1}{4}$ everywhere and hence one cannot infer stability from the theorem of Miles (1961) and Howard (1961). Nevertheless, the corresponding density distribution is statically stable and, intuitively, one might have thought that it would tend to have a further stabilizing effect on plane Couette flow, which is known to be stable in the homogeneous case.

Huppert's (1973) inviscid analysis of the problem, however, shows that this expectation is false. He found that the flow is unstable if the overall Richardson number† J_H is larger than $\frac{1}{4}$. More generally, he found (see his figure 2) that in the α, J_H plane there are regions of stability, regions of instability with $c_r = 0$ and regions of instability with $c_r \neq 0$. When these results are considered as the inviscid limit of the full viscous problem, it is clear that the modal structure of this problem for large values of the Reynolds number must differ substantially from that found in part 1. In this paper, therefore, we shall consider some aspects of the full viscous problem for stratified plane Couette flow with $N^2(z) = z^2$. For this problem it has not yet been possible to obtain any

† Huppert's analysis was on the interval $|z| \leqslant \pi$ whereas the present analysis is on the interval $|z| \leqslant 1$. Thus, if we let α_H, c_H and J_H denote the parameters appearing in his analysis, then they are related to the corresponding parameters defined in part 1 by $\alpha_H = \alpha/\pi$, $c_H = \pi c$ and $J_H = Ri/\pi^2$.

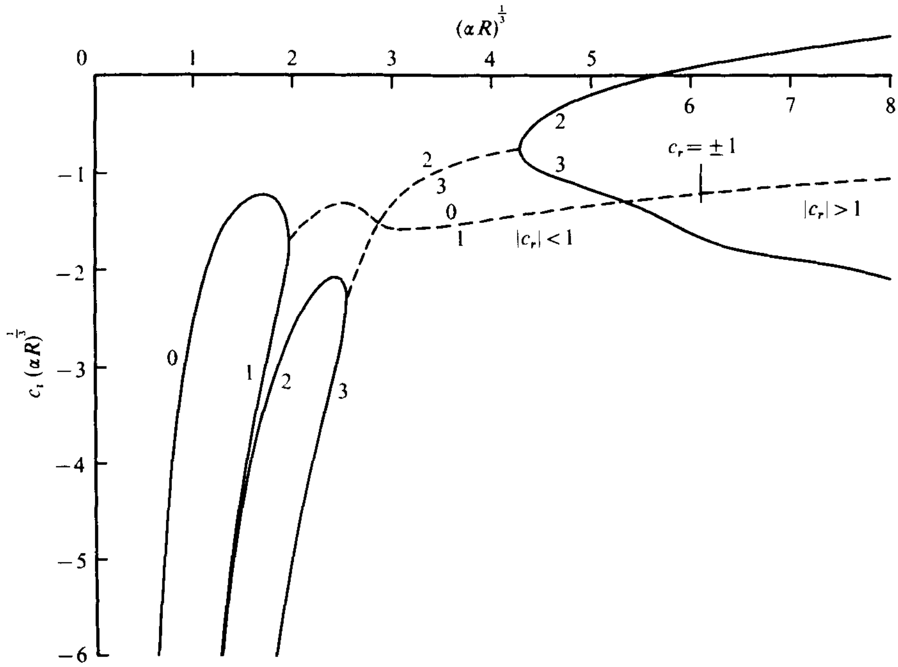

FIGURE 1. The modal structure for $J_H = \frac{1}{2}$ with $\alpha = 0$ but αR finite. ——, symmetric modes with $c_r = 0$; ---, asymmetric modes with $c_r \neq 0$.

simple analytical approximations of the type obtained in part 1. The present results were therefore obtained entirely by direct numerical methods and throughout the Prandtl number has been taken to be one.

For large values of the Reynolds number, the present results are in complete agreement with Huppert's inviscid analysis. In addition, the curves of neutral stability are found to be somewhat similar to those for homogeneous flows in a channel with an inflexion point and the minimum critical Reynolds numbers are correspondingly low. In an attempt to understand the basic physical mechanism of the instability we have also considered the Reynolds stress distribution and the vertical buoyancy flux for one typical value of J_H. These results show that there is a large transfer of energy from the basic flow to the disturbance flow (as compared, for example, with homogeneous plane Poiseuille flow) but a precise description of the physical mechanism involved still remains somewhat elusive.

2. The modal structure for $\alpha = 0$ and $J_H = \frac{1}{2}$

For $\alpha = 0$ and $J_H = \frac{1}{2}$, Huppert's inviscid analysis predicts the existence of one unstable mode with $c_r = 0$. This result is in marked contrast to the results obtained in part 1, where it was shown that, as $\alpha R \to \infty$, all modes are stable with $c_r \neq 0$. Accordingly, we first considered the modal structure of the viscous problem for $\alpha = 0$ (with αR finite) and $J_H = \frac{1}{2}$. The results for the first group of four modes are shown in figure 1.

As $\alpha R \to 0$, $c_r = 0$ and c_i is independent of both U and N^2. In this limit, therefore, the modes have the same behaviour as in part 1. For finite values of αR, however, their

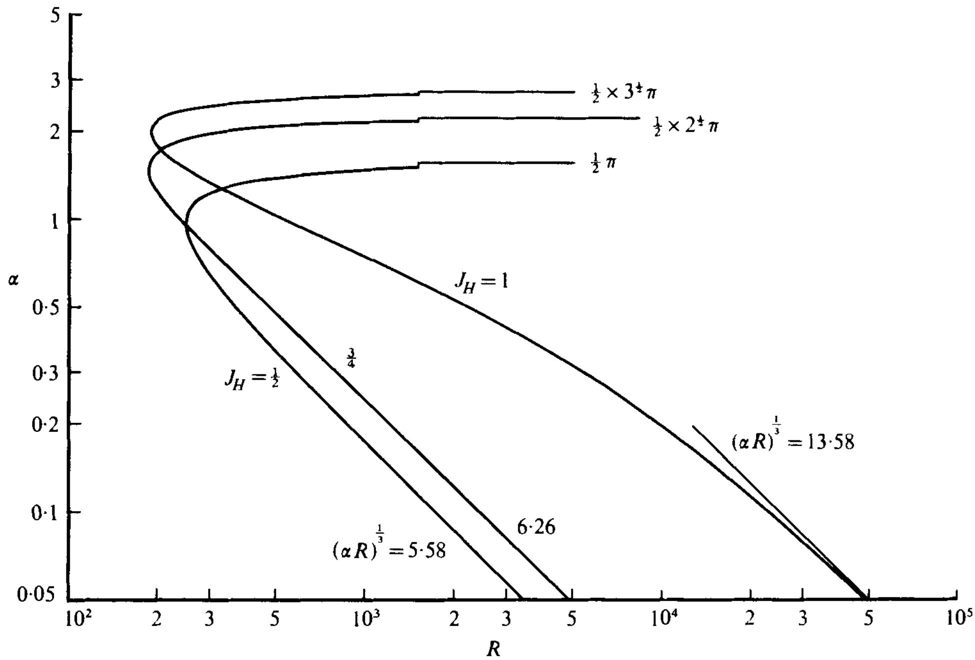

FIGURE 2. The curves of neutral stability for $J_H = \frac{1}{2}, \frac{3}{4}$ and 1. The asymptotes to the lower branches of these curves are given by $(\alpha R)^{\frac{1}{3}} = 5{\cdot}58$, $6{\cdot}26$ and $13{\cdot}58$ respectively.

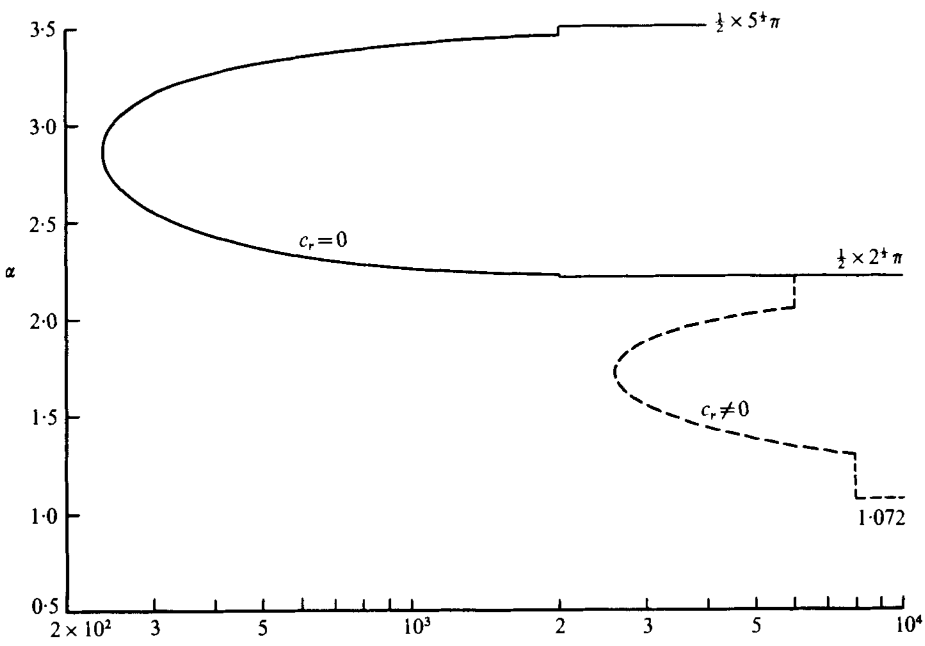

FIGURE 3. The curves of neutral stability for $J_H = \frac{3}{4}$.

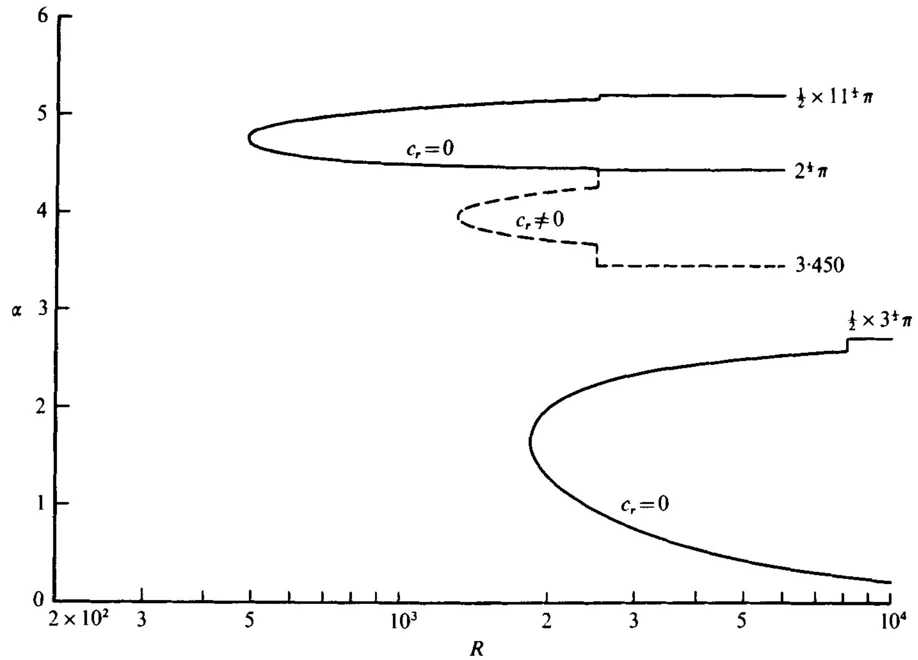

FIGURE 4. The curves of neutral stability for $J_H = 3$. The asymptote to the lower branch is given by $(\alpha R)^{\frac{1}{3}} = 12{\cdot}98$.

behaviour is dramatically different. The modes labelled 0 and 1 can be identified as temperature modes when $\alpha R \to 0$. When $\alpha R \to \infty$, however, they become a pair of travelling modes with $c_i \uparrow 0$ and c_r lying outside the range of U. Modes of the latter type presumably correspond to internal gravity waves and could be studied directly from the Taylor–Goldstein equation. Similarly, the modes labelled 2 and 3 can be identified as velocity modes when $\alpha R \to 0$ and it is one of these modes which becomes unstable for large values of αR. The values of $(\alpha R)^{\frac{1}{3}}$ at which $c_i = 0$ correspond, as shown in the following section, to the asymptote to the lower branch of the curve of neutral stability for $J_H = \frac{1}{2}$.

These results thus confirm one aspect of Huppert's analysis but they also raise an important question concerning the dependence of the minimum critical Reynolds number on J_H. To answer this question we must now consider the curves of neutral stability for some typical values of J_H.

3. The curves of neutral stability

According to Huppert's results, when $\frac{1}{4} < J_H \leqslant 1$ we would expect just one curve of neutral stability (for each value of J_H) along which $c_r = 0$, and this expectation is fully confirmed by the results shown in figure 2 for $J_H = \frac{1}{2}, \frac{3}{4}$ and 1. The asymptotes to the lower branches of these curves are simply the values of $(\alpha R)^{\frac{1}{3}}$ for which both $\alpha = 0$ and $c = 0$ as shown, for example, in figure 1. Before commenting further on these results, let us consider briefly the situation when $J_H > 1$. According to the inviscid theory, when $J_H > 1$ we would expect two or more curves of neutral stability and this expectation is again confirmed by the results shown in figures 3 and 4 for two fairly typical values of J_H.

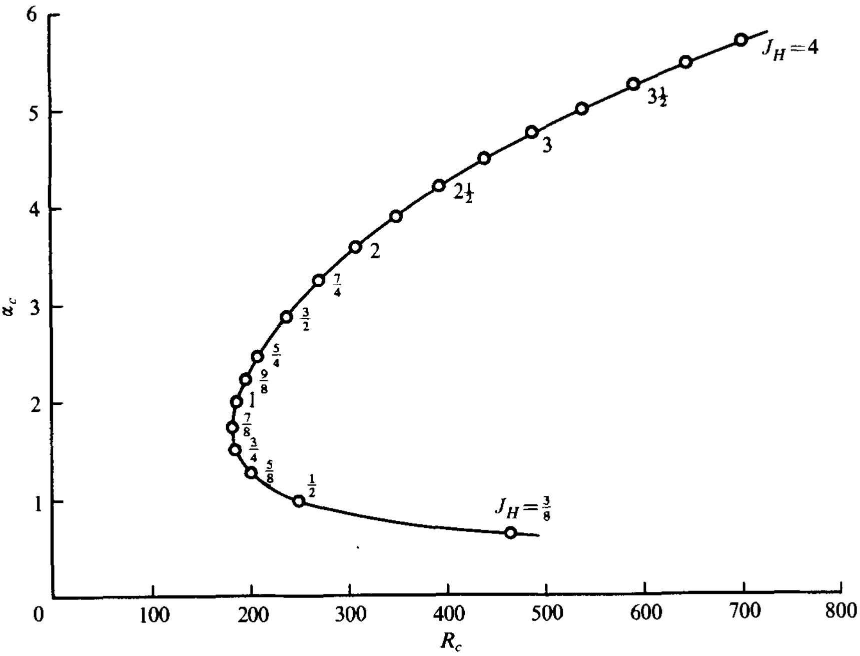

FIGURE 5. The variation of α_c and R_c with J_H. Along this curve $c_r = 0$.

The curves of neutral stability shown in figure 2 are similar to those found for flows of a homogeneous fluid in a channel with one inflexion point and the minimum critical Reynolds numbers for them are correspondingly low. This can be seen even more clearly in figure 5, which also shows that there is an overall minimum value of R_c for which

$$J_H = 0{\cdot}8416, \quad \alpha_c = 1{\cdot}674, \quad R_c = 182{\cdot}8.$$

Such a low value of R_c is really quite surprising when compared, for example, with plane Couette or plane Poiseuille flow of a homogeneous fluid, for which $R_c = \infty$ and 5772·2 respectively, and shows that a statically stable stratification can sometimes provide an efficient mechanism, which is not present in the flow of a homogeneous fluid, for transferring energy from the basic flow to the disturbance flow.

4. The Reynolds stress and the vertical buoyancy flux

To study the transfer of energy from the basic flow to the disturbance flow, we consider the energy equations for two-dimensional disturbances in a stratified parallel shear flow, which, in dimensionless form, are given by

$$\frac{d}{dt}\iint \tfrac{1}{2}(u'^2+w'^2)\,dx\,dz = -\iint u'w'\frac{dU}{dz}\,dx\,dz + Ri\iint \sigma' w'\,dx\,dz - \frac{1}{R}\iint\left(\frac{\partial w'}{\partial x}-\frac{\partial u'}{\partial z}\right)^2 dx\,dz \quad (4.1)$$

and

$$\frac{d}{dt}\iint \tfrac{1}{2}\sigma'^2\,dx\,dz = -\iint \sigma' w' N^2(z)\,dx\,dz - \frac{1}{RP}\iint\left\{\left(\frac{\partial\sigma'}{\partial x}\right)^2+\left(\frac{\partial\sigma'}{\partial z}\right)^2\right\}dx\,dz. \quad (4.2)$$

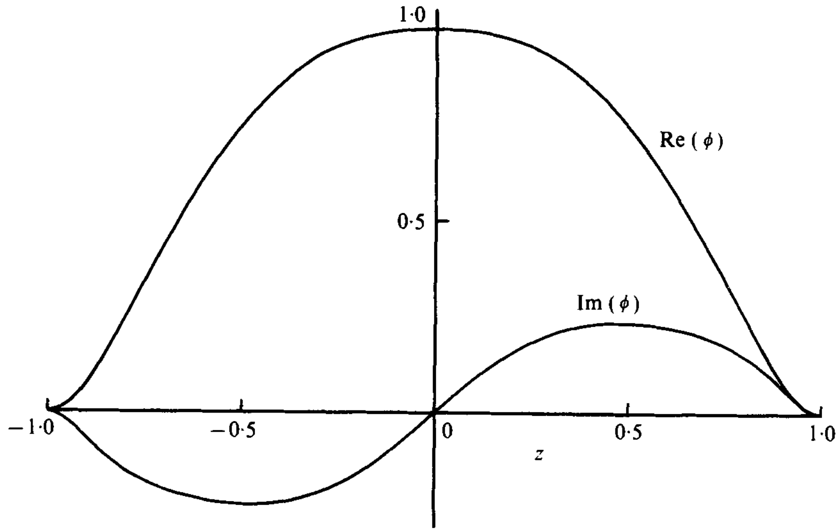

FIGURE 6. The eigenfunction ϕ for $J_H = \frac{1}{2}$, $\alpha = 0{\cdot}9630$ and $R = 249{\cdot}9$.

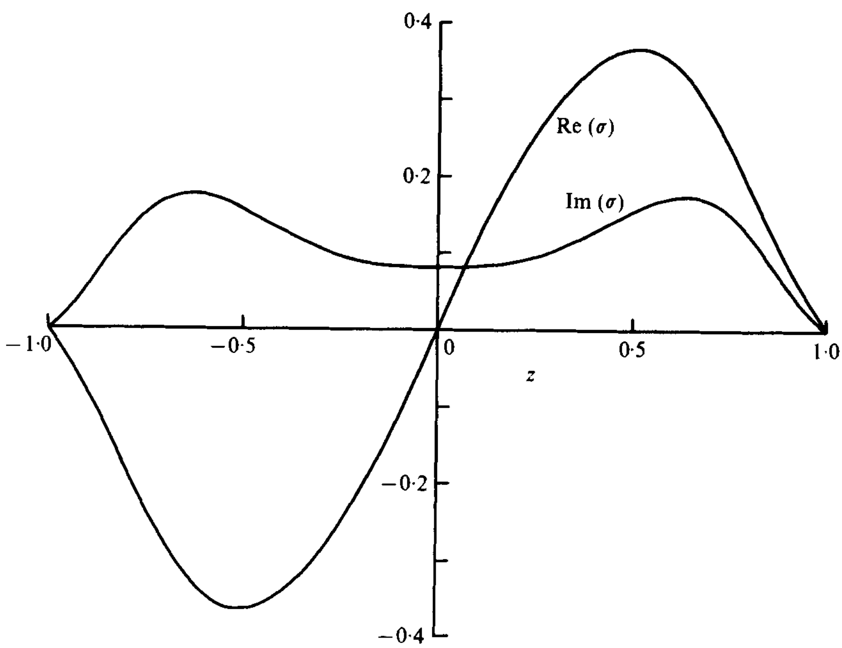

FIGURE 7. The eigenfunction σ for $J_H = \frac{1}{2}$, $\alpha = 0{\cdot}9630$, and $R = 249{\cdot}9$.

The (dimensionless) Reynolds stress and vertical buoyancy flux are then defined in the usual way by

$$\tau = -\langle u'w' \rangle, \quad b = \langle \sigma' w' \rangle, \tag{4.3}$$

where the angle brackets denote an average over one wavelength. In terms of the Reynolds stress function $S(z)$ and the vertical buoyancy flux function $B(z)$, we have

$$\{\tau, b\} = \tfrac{1}{2}\alpha\{S(z), B(z)\} \exp(2\alpha c_i t), \tag{4.4}$$

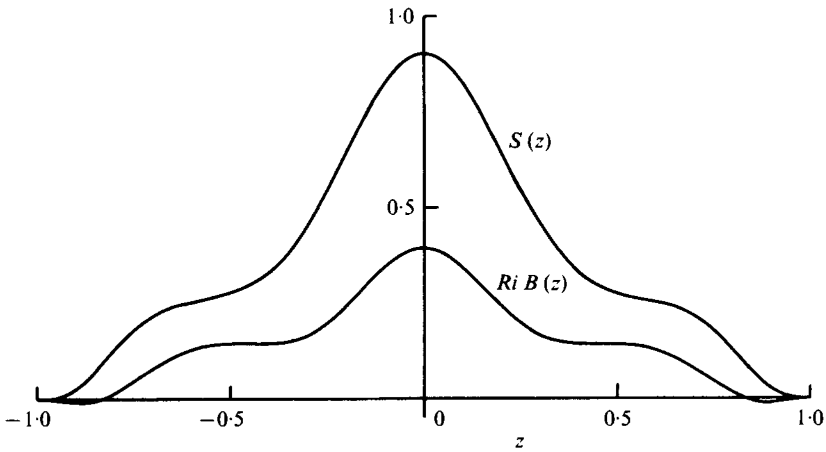

FIGURE 8. The Reynolds stress function $S(z)$ and the vertical buoyancy flux function $B(z)$ for $J_H = \frac{1}{2}$, $\alpha = 0{\cdot}9630$ and $R = 249{\cdot}9$.

where
$$S(z) = \phi_r \phi_i' - \phi_i \phi_r' \tag{4.5}$$
and
$$B(z) = \sigma_r \phi_i - \sigma_i \phi_r = Ri^{-1}\{(L_4\phi)_r \phi_i - (L_4\phi)_i \phi_r\}. \tag{4.6}$$

The eigenfunctions ϕ and σ have been calculated for $J_H = \frac{1}{2}$, $\alpha = 0{\cdot}9630$ and $R = 249{\cdot}9$, and these results are shown in figures 6 and 7. The normalization of these eigenfunctions was fixed by setting $\phi(0) = 1$. From these results it was then possible to compute the Reynolds stress function $S(z)$ and the vertical buoyancy flux function $B(z)$, and these results are shown in figure 8. Equation (4.1) shows that the sign of the Reynolds stress is such that there is a large transfer of energy from the basic flow to the velocity disturbances. This might have been anticipated in view of the low value of the minimum critical Reynolds number for this flow but it is in marked contrast, for example, to the case of plane Poiseuille flow of a homogeneous fluid, for which the Reynolds stress is small except near the critical points, and, in addition, its maximum value is smaller than $S(0)$ in figure 8 by about a factor of 10 (see Stuart 1963, figure IX.8). The vertical buoyancy flux also makes a significant contribution to the growth of the velocity disturbances but it has a much smaller effect, owing to the factor $N^2(z)$, on the temperature disturbance and has the same sign as the dissipative term in (4.2).

5. Discussion

The circumstances under which a statically stable stratification can have a destabilizing effect on shear flows of an inviscid fluid are quite varied, as the examples discussed by Huppert (1973) and Howard & Maslowe (1973) clearly show. The present results for Huppert's problem further show that when the effects of viscosity and thermal conductivity are both included the stratification can still have a strongly destabilizing effect. The inviscid stability characteristics found by Howard & Maslowe for plane Couette flow with $N^2(z) = 1 + a^2z^2$ are quite different since this flow is stable if $Ri > \frac{1}{4}$ but is unstable if $0 < Ri < \frac{1}{4}$ and a^2 is sufficiently large, and it would clearly be of interest to study the effects of viscosity and thermal conductivity on such a flow.

We had hoped to be able to give some analytical results to complement the present numerical results. Some progress has been made in the case of neutral stability for the standing modes, for which $c = 0$, but the analysis is substantially more difficult than that given in part 1. The major difficulty is associated with the fact that when $N^2(z) = z^2$ equation (2.7) of part 1 cannot be factorized even when $\alpha = 0$ with αR finite. As a result, uniform asymptotic approximations to the solutions of that equation cannot be obtained by a simple application of existing methods.

It would also be of interest to study the spectrum of internal gravity waves and its relationship to the velocity and temperature modes as $\alpha R \to \infty$. A simple analytical solution does not appear to be possible in this case, even when $\alpha = 0$, but the general results obtained by Banks, Drazin & Zaturska (1976) strongly suggest that the spectrum of internal gravity waves for this problem is qualitatively similar to that found in part 1.

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REFERENCES

BANKS, W. H. H., DRAZIN, P. G. & ZATURSKA, M. B. 1976 *J. Fluid Mech.* **75**, 149.

DAVEY, A. & REID, W. H. 1977 *J. Fluid Mech.* **80**, 509.

HOWARD, L. N. 1961 *J. Fluid Mech.* **10**, 509.

HOWARD, L. N. & MASLOWE, S. A. 1973 *Boundary-Layer Met.* **4**, 511.

HUPPERT, H. E. 1973 *J. Fluid Mech.* **57**, 361.

MILES, J. W. 1961 *J. Fluid Mech.* **10**, 496.

STUART, J. T. 1963 In *Laminar Boundary Layers* (ed. L. Rosenhead), pp. 492–579. Oxford: Clarendon Press.